# The Localized Energy Distribution of Dark Energy Star Solutions

Paul Halpern

and

Michael Pecorino

Department of Mathematics, Physics and Statistics,

University of the Sciences in Philadelphia,

600 S. 43rd St.,

Philadelphia, PA. 19104, USA

## ABSTRACT

We examine the question of energy localization for an exact solution of Einstein's equations with a scalar field corresponding to the phantom energy interpretation of dark energy. We apply three different energy-momentum complexes, the Einstein, Papapetrou and Møller prescriptions, to the exterior metric and determine the energy distribution for each. Comparing the results, we find that the three prescriptions yield identical energy distributions.

## 1. Introduction

The question of energy localization has been a pressing issue in general relativity since Einstein formulated his field equations. Einstein sought to include the energy and momenta of gravitational fields along with those of matter and non-gravitational fields as a well-defined locally conserved quantity. To that end, he proposed an energy-momentum complex that follow conservation laws for those quantities (Einstein 1915).

One concern about the Einstein energy-momentum is that its value depends on the coordinate system used. Specifically, it favors the use of quasi-Cartesian (perturbations of a flat background) coordinates. Accordingly, it does not transform as a tensor. Because it is asymmetric in its indices, it does not conserve angular momentum.

To address these issues, a number of other theorists have proposed alternative definitions for energy-momentum complexes. These include prescriptions by Papapetrou ( 1948), Landau and Lifshitz ( 1962), Weinberg ( 1972) and others, each designed to conserve angular momentum in addition to energy-momentum. A prescription by Møller ( 1958, 1961) offers the special advantage of being coordinate-system independent.

While at first the multiplicity of energy-momentum complexes discouraged theorists from using them, important results in the 1990s suggested an underlying unity that served to revive interest. In 1990, Virbhadra found consistency in the application of distinct energy-momentum prescriptions to the same metric (Virbhadra 1990). Six years later, a paper by Aguirregabiria, Chamorro and Virbhadra revealed that all metrics of the Kerr-Schild class, including the Schwarzchild, Reissner-Nordström and other solutions, yield identical energy and momentum distributions for a variety of prescriptions (Aguirregabiria, et. al. 1996(@). Then in 1999, a paper by Virbhadra indicated the energy-momentum complexes of Einstein, Papapetrou, Weinberg, and Landau and Lifshitz produced similar results for metrics more general than the Kerr-Schild class if they are constructed with Kerr-Schild

Cartesian coordinates (Virbhadra 1999).

There have subsequently been many attempts to apply the various energy-momentum complexes to black holes and related astronomical objects. In recent years, for example, Radinschi, along with various colleagues, has investigated the energy-momentum distribution for stringy black holes (Radinschi and Yang 2005; Radinschi and Ciobanu 2006), Hoŕava-Lifshitz black holes (Radinschi, et. al. 2011) , for charged black holes in generalized dilaton-axion gravity (Radinschi, et. al. 2010), for asymptotically de Sitter spacetimes (Radinschi 2011), and in a Schwarzschild-quintessence spacetime (Radinschi, et. al. 2012). Vagenas has examined the energy distribution in the dyadosphere of a Reissner-Nordström black hole (Vagenas 2006). Berman has looked at the energy and angular momentum of dilation black holes (Berman 2008). Sharif has examined the energy of rotating spacetimes in teleparallel gravity (Sharif and Jawad 2011). Halpern has investigated the energy distribution of a charged black hole with a minimally coupled scalar field (Halpern 2008).

In this paper, we consider yet another scenario: a static, electrically-neutral, spherically-symmetric massive object (such as a star or black hole) embedded in a space filled with phantom energy. Phantom energy is an especially potent type of dark energy proposed by Caldwell ( 2002) that possesses an equation of state parameter $w < -1$. It and other forms of dark energy have been hypothesized as possible agents for the acceleration of the universe, as recorded in supernova surveys (Riess, et. al. 1998; Perlmutter, et. al. 1999).

The interior and exterior metrics of a dark energy star were calculated by Yazadjiev ( 2011). The metrics depend on a mass parameter $m$, the radius $r$ and an additional parameter $\beta$. From these, Yazadjiev determined an overall mass $M$ (that includes the rest mass as well as dark energy) and a dark charge $D$, through the following relationships:

$$M = \cosh\beta \; m \tag{1}$$

$$D = \sinh\beta \; m \tag{2}$$

Note that $\beta = 0$ corresponds to the situation of a conventional Schwarzschild solution without phantom energy. The conventions that $c = 1$ and $G = 1$ are followed.

Applying the prescriptions of Einstein, Papapetrou and Møller to the exterior metric derived by Yazadjiev we will examine the localized energy of the dark energy star.

## 2. Applying Einstein's prescription to dark energy stars

We now apply Einstein's prescription to Yazadjiev's dark energy star solution. The exterior metric has the following form:

$$
\begin{aligned}
ds^2 = {} & -(1 - 2m/r)^\kappa dt^2 \\
& + (1 - 2m/r)^{1-\kappa} [\frac{dr^2}{1 - \frac{2m}{r}} \\
& + r^2 (d\theta^2 + \sin\theta^2 d\phi^2)]
\end{aligned}
\tag{3}
$$

where:

$$\kappa = \cosh\beta = \frac{M}{\sqrt{M^2 - D^2}} \tag{4}$$

Einstein's prescription defines the local energy-momentum density as:

$$\theta_i{}^k = \frac{1}{16\pi} H_i{}^{kl}{}_{,l} \tag{5}$$

where the superpotentials $H_i{}^{kl}$ are given by:

$$H_i{}^{kl} = \frac{g_{in}}{\sqrt{-g}} [-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m} \tag{6}$$

This complex has the antisymmetric property that:

$$H_i{}^{kl} = -H_i{}^{lk} \tag{7}$$

The energy-momentum components can be found by integrating the energy-momentum density over the volume under consideration:

$$P_i = \int \int \int \theta_i{}^0 dx^1 dx^2 dx^3 \tag{8}$$

Through Gauss's theorem we can express this as a surface integral:

$$P_i = \frac{1}{16\pi} \int \int H_i{}^{0\alpha} \mu_\alpha dS \tag{9}$$

where $\mu_\alpha$ is the outward unit vector normal to the spherical surface element:

$$dS = r^2 \sin\theta \, d\theta \, d\phi \tag{10}$$

The localized energy $E = P_0$ can thereby be expressed as:

$$P_0 = \frac{1}{16\pi} \int \int H_0{}^{0\alpha} \mu_\alpha dS \tag{11}$$

Therefore, the relevant superpotentials to determine the localized energy are $H_0{}^{0\alpha}$ with $\alpha$ ranging from 1 to 3.

Taking $i = k = 0$ we find that (6) reduces to:

$$H_0{}^{0l} = \frac{g_{0n}}{\sqrt{-g}}[-g(g^{0n}g^{lm} - g^{ln}g^{0m})]_{,m} \tag{12}$$

Note that $g_{0n} = 0$ for all values of $n$ except $n = 0$ Therefore, the only non-zero components of (12) are the ones for which $n = 0$, namely:

$$H_0{}^{0l} = \frac{g_{00}}{\sqrt{-g}}[-g(g^{00}g^{lm} - g^{l0}g^{0m})]_{,m} \tag{13}$$

We substitute the metric (3) into expression (13) and find that the relevant superpotential values are:

$$H_0{}^{01} = \frac{4\kappa mx}{r^3} \tag{14}$$

$$H_0{}^{02} = \frac{4\kappa my}{r^3} \tag{15}$$

$$H_0{}^{03} = \frac{4\kappa mz}{r^3} \tag{16}$$

We insert the super potentials (14-16) into the double integral (11). Evaluating this double integral over the full coordinate range, we find the total energy of s solution within a sphere of radius $r$ to be:

$$E = P_0 = \kappa m = M \tag{17}$$

It is interesting that the localized energy precisely matches the dark energy star's total mass $M$. This indicates that the Einstein complex well-encompasses the complete energy of the star, including the energy associated with its rest mass $m$ as well as its dark energy. Note that if the dark charge $D$ is zero, $\kappa$ becomes 1 and the localized energy reverts to the Schwarzschild value of $m$.

## 3.   Determining the energy by use of Papapetrou's prescription

We now turn to a second method for determining the local energy, Papapetrou's prescription. Unlike Einstein's prescription, Papapetrou's energy-momentum complex offers

the advantage of being symmetric in its indices. Hence it permits the precise definition of local conservation laws.

The Papapetrou energy-momentum complex is defined as:

$$\Omega^{ik} = \frac{1}{16\pi} N^{ikab}{}_{,ab} \tag{18}$$

$$P_i = \int \int \int \Omega^{i0} dx^1 dx^2 dx^3 \tag{19}$$

where the functions $N^{ikab}$ are given by:

$$N^{ikab} = \sqrt{-g} \left[ g^{ik}\eta^{ab} - g^{ia}\eta^{kb} + g^{ab}\eta^{ik} - g^{kb}\eta^{ia} \right] \tag{20}$$

and the $\eta^{ab}$ terms represent the components of a Minkowski metric of signature $-2$.

Again, we use Gauss's theorem to express the total energy as a surface integral:

$$E = \frac{1}{16\pi} \int \int \chi^{00\alpha} \mu_\alpha dS \tag{21}$$

with the superpotentials $\chi^{00\alpha}$ defined as:

$$\chi^{00\alpha} = N^{00k\alpha}_{,k} \tag{22}$$

We determine the relevant values of the superpotentials to be:

$$\chi^{001} = \frac{4\kappa m x}{r^3} \tag{23}$$

$$\chi^{002} = \frac{4\kappa my}{r^3} \tag{24}$$

$$\chi^{003} = \frac{4\kappa mz}{r^3} \tag{25}$$

Substituting equations (20-22) into (18) yields:

$$E = P_0 = \kappa m = M \tag{26}$$

This is identical to the expression obtained using Einstein's prescription. It is instructive to see that both complexes produce the same result.

### 4. Determining the energy by use of the Møller prescription

We now turn to a third method for determining the local energy, the Møller prescription, which has the marked advantage of being coordinate-system-independent. The complex defined by Møller can be expressed as:

$$\Xi_i^k = \frac{1}{8\pi} \chi_{i,p}^{kp} \tag{27}$$

where:

$$\chi_i^{kl} = \sqrt{-g} \left[ g_{ip,q} - g_{iq,p} \right] g^{kq} g^{lp} \tag{28}$$

Inserting the metric components for Yazadjiev's solution (3) we obtain:

$$\chi_0{}^{01} = \frac{4\kappa mx}{r^3} \tag{29}$$

$$\chi_0{}^{02} = \frac{4\kappa m y}{r^3} \tag{30}$$

$$\chi_0{}^{03} = \frac{4\kappa m z}{r^3} \tag{31}$$

One more time, we employ Gauss's theorem to express the total energy as a surface integral:

$$E = \frac{1}{16\pi} \int \int \chi_0{}^{0\alpha} \mu_\alpha dS \tag{32}$$

Integrating over the full range of coordinates, we find the total energy using the Møller complex to be:

$$E = \kappa m = M \tag{33}$$

This is identical to the expression obtained using Einstein's and Papapetrou's prescriptions.

## 5.   Conclusion

We have determined the localized energy distribution for Yazadjiev's solution representing a static, electrically-neutral, spherically-symmetric massive object with phantom energy. In applying the Einstein, Papapetrou and Møller energy-momentum complexes to Yazadjiev's metric, we have found that each yields an identical localized energy equal to the mass $M$. This extends earlier results for Kerr-Schild objects such as the Schwarzschild solution to an interesting case that could bear upon the dark energy question. Our findings help broaden the applicability of energy-momentum complexes, augmenting their use in general relativity as consistent, well-defined ways of describing the local distribution of energy.

# REFERENCES

Aguirregabiria, J. M., Chamorro, A. and K. Virbhadra, K.S. (1996) Gen. Rel. Grav. 28, 1393.

Berman, M.S. (2008) Revista Mex. Astron. Astrof. 44, 285.

Caldwell, R. (2002) Phys. Lett. B 545, 23.

Einstein, A. (1915) Preuss. Akad. Wiss. Berlin 47, 778.

Halpern, P. (2008) Astrophys. and Space Sci. 313, 4, 357.

Landau, L.D. and Lifshitz, E.M. (1962) The Classical Theory of Fields, Pergamon Press, Oxford, 341.

M$\phi$ller, C. (1958) Ann. Phys. 4, 347.

M$\phi$ller, C. (1961) Ann. Phys. 12, 118.

Papapetrou, A. (1948) Proc. R. Ir. Acad. A 52, 11.

Perlmutter, S. et al. [Supernova Cosmology Project Collaboration] (1999) Astrophys. J. 517, 565.

Radinschi, I. (2011) Cent. Eur. Jour. of Phys 9, 5, 1173.

Radinschi, I. and Ciobanu, B. (2006) "Weinberg Energy-Momentum Complex for a Stringy Black Hole Solution," gr-qc/0608029.

Radinschi, I., Grammenos, T. and Spanou, A.( 2012) "Distribution of Energy-Momentum in a Schwarzschild-Quintessence Space-time Geometry," http://arxiv.org/abs/1204.1663.

Radinschi, I., Rahaman F. and Banerjee, A. (2011) Int. J. Theor. Phys. 50, 9, 2906.

Radinschi, I., Rahaman, F. and Ghosh, A. (2010) Int. J. Theor. Phys 49, 943.

Radinschi I. and Yang, I.C. "On the Energy of String Black Holes," (2005) New Developments in String Theory Research, ed. Susan A. Grece, New York: Nova Science.

Riess, A.G., et al. [Supernova Search Team Collaboration] (1998) Astron. J. 116, 1009.

Sharif, M. and Jawad, A. (2011) Astrophys. and Space Sci. 331, 1, 321.

Vagenas, E.C. (2006) Mod. Phys. Lett. A21, 1947.

Virbhadra, K.S. (1990) Phys. Rev. D41, 1086.

Virbhadra, K.S. (1999) Phys. Rev. D60, 104041.

Weinberg, S. (1972) Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley, New York, 165.

Yazadjiev, S. (2011) Phys. Rev. D. 83, 127501.